

# Stabilizing matter and gauge fields localized on walls

Masato Arai<sup>1,2,\*</sup>, Filip Blaschke<sup>2,\*\*</sup>, Minoru Eto<sup>3,\*\*\*</sup>, and Norisuke Sakai<sup>4,\*\*\*\*</sup>

<sup>1</sup>*Fukushima National College of Technology, Iwaki, Fukushima 970-8034, Japan*

<sup>2</sup>*Institute of Experimental and Applied Physics, Czech Technical University in Prague, Horská 22, 128 00 Prague 2, Czech Republic, and Institute of Physics, Silesian University in Opava, Bezručovo nám. 1150/13, 746 01 Opava, Czech Republic*

<sup>3</sup>*Department of Physics, Yamagata University, Yamagata 990-8560, Japan*

<sup>4</sup>*Department of Mathematics, Tokyo Woman's Christian University, Tokyo 167-8585, Japan*

*E-mail: masato.arai@fukushima-nct.ac.jp, \*\* Email: filip.blaschke@fpf.slu.cz,*

*\*\*\* Email: meto@sci.kj.yamagata-u.ac.jp \*\*\*\* Email: norisuke.sakai@gmail.com*

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Both non-Abelian gauge fields and minimally interacting massless matter fields are localized on a domain wall in the five-dimensional spacetime. Field-dependent gauge coupling naturally gives a position-dependent coupling to localize non-Abelian gauge fields on the domain wall. An economical field content allows us to eliminate a moduli for a instability, and to demonstrate the positivity of the position-dependent coupling in the entire moduli space. Effective Lagrangian similar to the chiral Lagrangian is found with a new feature of different coupling strengths for adjoint and singlet matter that depend on the width of the domain wall.

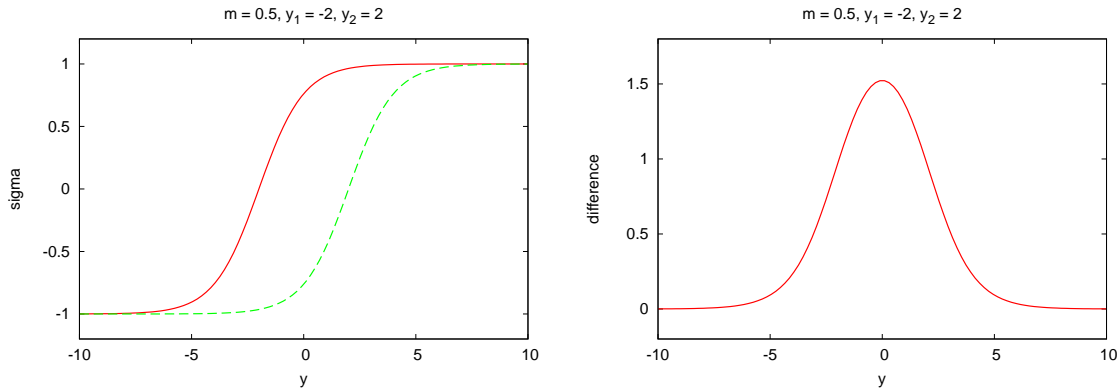
## 1. Introduction

Localization of massless gauge fields on a domain wall with  $3 + 1$  dimensional world volume has been a long-standing problem to achieve the dynamical compactification in the brane-world scenario [1]. If the gauge symmetry is unbroken only inside the domain wall and broken outside, the gauge field inevitably acquires a mass proportional to the inverse of the width of the wall [2–4]. The localization of gauge fields can be achieved if the gauge theory is in the confining phase outside of the domain wall [2, 3]. This requirement can be translated into a position-dependent gauge coupling [5–7].

The supersymmetric gauge theories in  $4 + 1$  spacetime dimensions allows a cubic coupling [8] between adjoint scalar fields  $\Sigma^\alpha$  and gauge field strengths  $F_{MN}^\beta$

$$\mathcal{L} \sim C_{\alpha\beta\gamma} \Sigma^\alpha F_{MN}^\beta F^{\gamma MN}, \quad (1.1)$$

with coupling constants  $C_{\alpha\beta\gamma}$ . In Ref.[9], a domain wall solution is chosen so that the scalar field  $\Sigma$  is positive inside and vanishes asymptotically outside of the domain wall. In this way, non-Abelian gauge fields have been localized on the domain wall. More recently a method to localize non-Abelian gauge fields together with minimally interacting matter fields on domain walls has been introduced and a particular model has been presented in five-dimensional spacetime [10], by gauging the unbroken global symmetry associated to the degenerate scalar fields to form the domain wall [11]. This mechanism can be viewed as a field theoretical realization of D-branes, and provides a step towards a realistic model of branes as soliton solutions of higher dimensional field theories.



**Fig. 1** Position-dependent gauge coupling (right panel) is given as a difference  $\sigma_1 - \sigma_2$  of two separated kinks (left panel).

In Ref. [9], two real scalar fields  $\sigma_1$  and  $\sigma_2$  were introduced to form the usual kink profile in the extra-dimensional coordinate  $y$  as shown in the left panel of Fig. 1, and the cubic coupling in Eq.(1.1) is chosen as

$$\mathcal{L}_{\text{cubic}} \sim -(\sigma_1 - \sigma_2) \text{Tr}[\tilde{G}_{MN} \tilde{G}^{MN}], \quad (1.2)$$

where  $\tilde{G}_{MN}$  is the non-Abelian gauge field strength to be localized on the domain wall. In this way, one achieved the desired profile of the position-dependent coupling which is positive and vanishes asymptotically outside of the domain wall as shown in the right panel of Fig.1. More recently, matter fields have also been localized on the domain wall interacting minimally with the localized gauge fields [10]. Although the model realized the localization mechanism in a simple setup, the width of the position-dependent coupling is a modulus and can become negative implying an instability of the gauge kinetic term.

Moduli of domain walls arise as remaining degrees of freedom of scalar fields constrained by gauge symmetry to form the domain wall. This observation prompts us to construct a model with smaller number of scalar fields with a different charge assignments in order to eliminate the unwanted modulus in the position-dependent coupling [9].

The purpose of this paper is to present models in  $4 + 1$  dimensional spacetime with domain wall solutions which allow stable localized gauge fields interacting minimally with localized matter fields. For any values of the moduli parameters, we show that the position-dependent gauge coupling is positive everywhere in the extra-dimensional coordinate  $y$ . In particular, its width is no longer a modulus and is fixed by the parameters of the theory. We thus find that the gauge kinetic term is stable for all values of moduli, namely for all possible domain wall configurations. As localized matter fields, we obtain scalar fields in singlet and adjoint representations of the gauge group. They are associated with the broken part of the global symmetry, and interact minimally with the localized gauge fields. The effective Lagrangian on the domain wall is also worked out, using the method in Ref.[12]. It resembles the chiral Lagrangian associated with the chiral symmetry breaking of QCD. A new feature of the effective Lagrangian is that the coupling strength of the adjoint matter fields is larger than that of the singlet matter fields. We examine generality of models with a stable kinetic term

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|          | $SU(N)_c$                      | $U(1)_1$ | $U(1)_2$ | $SU(N)_L$    | $SU(N)_R$    | $U(1)_A$ | mass            |
|----------|--------------------------------|----------|----------|--------------|--------------|----------|-----------------|
| $H_1$    | $\square$                      | 1        | 0        | $\square$    | $\mathbf{1}$ | 1        | $m\mathbf{1}_N$ |
| $H_2$    | $\square$                      | 1        | -1       | $\mathbf{1}$ | $\square$    | -1       | 0               |
| $H_3$    | $\mathbf{1}$                   | 0        | 1        | $\mathbf{1}$ | $\mathbf{1}$ | 0        | 0               |
| $\Sigma$ | $\text{adj} \oplus \mathbf{1}$ | 0        | 0        | $\mathbf{1}$ | $\mathbf{1}$ | 0        | 0               |
| $\sigma$ | $\mathbf{1}$                   | 0        | 0        | $\mathbf{1}$ | $\mathbf{1}$ | 0        | 0               |

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**Table 1** Quantum numbers of fields of the model for the domain wall.

for gauge fields to identify the class of models with desirable properties. We also discuss ways to build more realistic models for a phenomenology of the brane-world scenario.

The organization of the paper is as follows. Section 2 is devoted to construct a model in  $4+1$  spacetime dimensions with domain wall solutions which localizes matter and gauge fields. The position-dependent gauge coupling is shown to be positive. In Section 3, the low-energy effective Lagrangian is obtained. In Section 4, we examine the generality of models with the stable gauge kinetic term. Conclusion and discussion are in Section 5. Some details to derive the low-energy effective Lagrangian is given in Appendix A. In Appendix B, we summarize geometrical features of the three-flavor model.

## 2. A model with localized matter and gauge fields

In this section we first present a model allowing the domain wall solution with unbroken non-Abelian global symmetry [11]. As the second step, we introduce non-Abelian gauge fields for the unbroken symmetry. As the third step, we consider the cubic coupling [9] in Eq.(1.1) to localize non-Abelian gauge fields : we use expectation values of a singlet scalar field to give the position-dependent gauge coupling, whose positivity is demonstrated for any values of moduli parameters. A broken part of the global symmetry provides matter fields in singlet and adjoint representations of the localized gauge fields [10].

### 2.1. Lagrangian with global symmetry and domain wall solutions

Let us consider a five-dimensional  $SU(N)_c \times U(1)_1 \times U(1)_2$  gauge theory and  $N$  scalar fields  $H_1$  ( $H_2$ ) in the fundamental representation with the degenerate mass  $m$  ( $-m$ ), together with a singlet scalar field  $H_3$ , whose charge assignments are summarized in Tab. 1. Therefore we obtain global symmetry  $SU(N)_L \times SU(N)_R \times U(1)_A$ . In addition, we introduce adjoint and singlet scalars  $\Sigma$  and  $\sigma$  associated with the gauge group  $SU(N)_c \times U(1)_1$  and  $U(1)_2$ , respectively. We assume the following Lagrangian with the signature  $(+, -, -, -, -)$

$$\mathcal{L} = -\frac{1}{2g^2}\text{Tr}\left(G_{MN}G^{MN}\right) - \frac{1}{4e^2}F_{MN}F^{MN} + \frac{1}{g^2}\text{Tr}\left(D_M\Sigma\right)^2 + \frac{1}{2e^2}(\partial_M\sigma)^2$$

$$+ \text{Tr}|D_M H_1|^2 + \text{Tr}|(D_M - iA_M)H_2|^2 + |(\partial_M + iA_M)H_3|^2 - V, \quad (2.1)$$

$$V = \text{Tr}|(\Sigma - m\mathbf{1}_N)H_1|^2 + \text{Tr}|(\Sigma - \sigma\mathbf{1}_N)H_2|^2 + |\sigma H_3|^2$$

$$+ \frac{1}{4}g^2\text{Tr}\left(c_1\mathbf{1}_N - H_1H_1^\dagger - H_2H_2^\dagger\right)^2 + \frac{1}{2}e^2\left(c_2 + \text{Tr}(H_2H_2^\dagger) - |H_3|^2\right)^2. \quad (2.2)$$

The  $U(N)_c = SU(N)_c \times U(1)_1$  gauge coupling and gauge fields are denoted by  $g$  and an  $N \times N$  matrix  $W_M$  with  $M = 0, 1, 2, 3, 4$ . The  $U(1)_2$  gauge coupling and gauge field are

denoted by  $e$  and  $A_M$ . Covariant derivatives and field strengths are defined by

$$D_M H_{1,2} = \partial_M H_{1,2} + i W_M H_{1,2}, \quad D_M \Sigma = \partial_M \Sigma + i [W_M, \Sigma], \quad (2.3)$$

and  $G_{MN} = \partial_M W_N - \partial_N W_M + i [W_M, W_N]$ ,  $F_{MN} = \partial_M A_N - \partial_N A_M$ .

The global symmetry  $U_L \in SU(N)_L$ ,  $U_R \in SU(N)_R$ ,  $e^{i\alpha} \in U(1)_A$ , and the local gauge symmetry  $U_c \in U(N)_c$ ,  $e^{i\beta} \in U(1)_2$  act on the fields as

$$H_1 \rightarrow e^{i\alpha} U_c H_1 U_L, \quad H_2 \rightarrow e^{-i(\alpha+\beta)} U_c H_2 U_R, \quad H_3 \rightarrow e^{i\beta} H_3, \quad (2.4)$$

$$\Sigma \rightarrow U_c \Sigma U_c^\dagger, \quad \sigma \rightarrow \sigma. \quad (2.5)$$

Let us note that the Lagrangian (2.2) can be embedded into the five-dimensional  $\mathcal{N} = 1$  supersymmetric gauge theory (with 8 supercharges). This fact allows us to obtain the so-called Bogomol'nyi-Prasad-Sommerfield (BPS) domain wall solution as we will show in the next subsection. We stress, however, that this particular choice is merely to simplify calculations and is not required for our results to hold.

Without loss of generality, we can assume the mass parameter  $m$  to be positive. We also assume  $c_1 > 0$  and  $c_2 > 0$ . Then there exist  $N + 1$  discrete vacua with  $r = 0, 1, \dots, N$ , where scalar fields develop vacuum expectation value (VEV)

$$H_1 = \sqrt{c_1} \begin{pmatrix} \mathbf{1}_{N-r} & \\ & \mathbf{0}_r \end{pmatrix}, \quad H_2 = \sqrt{c_1} \begin{pmatrix} \mathbf{0}_{N-r} & \\ & \mathbf{1}_r \end{pmatrix}, \quad H_3 = \sqrt{c_2 + r c_1}, \quad (2.6)$$

$$\Sigma = m \begin{pmatrix} \mathbf{1}_{N-r} & \\ & \mathbf{0}_r \end{pmatrix}, \quad \sigma = 0. \quad (2.7)$$

Local gauge symmetry is completely broken and only a subgroup of global symmetry remains in these vacua. The breaking patterns in the  $r = 0$  and  $r = N$  vacua are

$$\begin{aligned} & U(N)_c \times SU(N)_L \times SU(N)_R \times U(1)_2 \times U(1)_A \\ & \xrightarrow{\text{0-th vacuum}} SU(N)_{L+c} \times SU(N)_R \times U(1)_{A+c}, \\ & \xrightarrow{\text{N-th vacuum}} SU(N)_{R+c} \times SU(N)_L \times U(1)_{A-c}. \end{aligned}$$

Let us consider domain wall solutions connecting  $N$ -th (0-th) vacuum at left (right) infinity  $y = -\infty$  ( $y = \infty$ ). Then the (coincident) domain wall solutions preserve the diagonal subgroup as the largest global symmetry  $SU(N)_{L+R+c}$ , providing the Nambu-Goldstone modes associated to the breaking of the global symmetry

$$\frac{SU(N)_L \times SU(N)_R \times U(1)_A}{SU(N)_{L+R+c}}. \quad (2.8)$$

We assume that fields depend only on extra-dimensional coordinate  $y$  and that all gauge fields except  $W_y$  and  $A_y$  vanish.

Thanks to the special choice of the potential, we can rewrite the energy density as

$$\begin{aligned}\mathcal{E} = & \frac{1}{g^2} \text{Tr} \left[ D_y \Sigma - \frac{g^2}{2} (c_1 \mathbf{1}_N - H_1 H_1^\dagger - H_2 H_2^\dagger) \right]^2 + \text{Tr} |D_y H_1 + (\Sigma - m \mathbf{1}_N) H_1|^2 \\ & + \frac{1}{2e^2} \left( \partial_y \sigma - e^2 (c_2 + \text{Tr}(H_2 H_2^\dagger - |H_3|^2)) \right)^2 + \text{Tr} |D_y H_2 + (\Sigma - (\sigma + i A_y) \mathbf{1}_N) H_2|^2 \\ & + |\partial_y H_3 + (\sigma + i A_y) H_3|^2 + c_2 \partial_y \sigma \\ & + \partial_y \text{Tr} \left[ c_1 \Sigma - H_1 H_1^\dagger (\Sigma - m \mathbf{1}_N) - H_2 H_2^\dagger (\Sigma - \sigma \mathbf{1}_N) \right].\end{aligned}\quad (2.9)$$

Thus, we obtain the Bogomol'nyi bound for the total energy (per unit world volume)  $E$

$$E = \int_{-\infty}^{\infty} dy \mathcal{E} \geq T = c_1 \int_{-\infty}^{\infty} dy \partial_y \text{Tr}(\Sigma) = N m c_1, \quad (2.10)$$

where  $T$  is the tension of the domain wall. This bound is saturated when the following BPS equations are satisfied

$$\partial_y H_1 + (\Sigma + i W_y - m \mathbf{1}_N) H_1 = 0, \quad (2.11)$$

$$\partial_y H_2 + \left( \Sigma + i W_y - (\sigma + i A_y) \mathbf{1}_N \right) H_2 = 0, \quad (2.12)$$

$$\partial_y H_3 + (\sigma + i A_y) H_3 = 0, \quad (2.13)$$

$$D_y \Sigma = \frac{1}{2} g^2 (c_1 \mathbf{1}_N - H_1 H_1^\dagger - H_2 H_2^\dagger), \quad (2.14)$$

$$\partial_y \sigma = e^2 (c_2 + \text{Tr}(H_2 H_2^\dagger - |H_3|^2)). \quad (2.15)$$

To use the moduli-matrix formalism [13–15], we introduce  $S(y) \in GL(N, \mathbb{C})$  and  $\psi(y) \in \mathbb{C}$

$$\Sigma + i W_y = S^{-1} \partial_y S, \quad \sigma + i A_y = \frac{1}{2} \partial_y \psi. \quad (2.16)$$

Then the matter part (2.11)-(2.13) can be solved by

$$H_1 = e^{m y} S^{-1} H_1^0, \quad (2.17)$$

$$H_2 = e^{\frac{1}{2} \psi} S^{-1} H_2^0, \quad (2.18)$$

$$H_3 = e^{-\frac{1}{2} \psi} H_3^0, \quad (2.19)$$

with complex constant  $N \times N$  moduli matrices  $H_1^0, H_2^0$  and a moduli constant  $H_3^0$ , which describe moduli of the solution. The rest of the BPS equations (2.14) and (2.15) turn into the master equations for gauge-invariant Hermitian fields  $\Omega \equiv S S^\dagger$  and  $\eta \equiv \text{Re}(\psi)$ .

$$\partial_y (\partial_y \Omega \Omega^{-1}) = \frac{1}{2} g^2 (c_1 \mathbf{1}_N - (e^{2m y} H_1^0 H_1^{0\dagger} + e^\eta H_2^0 H_2^{0\dagger}) \Omega^{-1}), \quad (2.20)$$

$$\frac{1}{2} \partial_y^2 \eta = e^2 (c_2 + e^\eta \text{Tr}(H_2^0 H_2^{0\dagger} \Omega^{-1}) - e^{-\eta} |H_3^0|^2). \quad (2.21)$$

Moduli matrices related by the following  $V$ -transformations give identical physical fields

$$(S, \psi, H_1^0, H_2^0, H_3^0) \rightarrow (V S, \psi + v, V H_1^0, V H_2^0 e^{-\frac{1}{2} v}, H_3^0 e^{\frac{1}{2} v}), \quad (2.22)$$

where  $V \in GL(N, \mathbb{C})$  and  $v \in \mathbb{C}$ . The equivalence class quotiented by this  $V$ -transformation defines the moduli space of domain walls. We can use this freedom to choose the form of the

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moduli matrices

$$H_1^0 = \sqrt{c_1} \mathbf{1}_N, \quad H_3^0 = \sqrt{c_2}. \quad (2.23)$$

Let us also decompose  $H_2^0$  as

$$H_2^0 = \sqrt{c_1} e^\phi U^\dagger, \quad (2.24)$$

where  $\phi$  is a Hermitian  $N \times N$  matrix and  $U$  is a unitary  $N \times N$  matrix. With this choice, the master equations (2.20) and (2.21) become

$$\partial_y(\partial_y \Omega \Omega^{-1}) = \frac{c_1}{2} g^2 (\mathbf{1}_N - \Omega_0 \Omega^{-1}), \quad (2.25)$$

$$\frac{1}{2} \partial_y^2 \eta = e^2 \left( c_2 + c_1 e^\eta \text{Tr}(e^{2\phi} \Omega^{-1}) - e^{-\eta} c_2 \right), \quad (2.26)$$

where  $\Omega_0 = e^{2my} \mathbf{1}_N + e^{2\phi} e^\eta$ .

No analytic solution of this system of the differential equations is known in general. However, one can study essential features of solutions, if one takes the strong gauge coupling limit  $g^2, e^2 \rightarrow \infty$ . Eqs. (2.25) and (2.26) reduce to a system of algebraic equations in this limit:

$$\Omega = e^{2my} \mathbf{1}_N + e^{2\phi} e^\eta, \quad (2.27)$$

$$c_2 = e^{-\eta} c_2 - c_1 e^\eta \text{Tr}(e^{2\phi} \Omega^{-1}). \quad (2.28)$$

It turns out that the effective theory describing massless excitations localized on the background solution of the equations of this system precisely coincides (at least in the lowest order of approximation) with the one obtained from (2.25) and (2.26) (see the detailed discussion in Ref. [10]). It is therefore sufficient just to study solutions of (2.27) and (2.28).

Without loss of generality we can assume that the moduli  $\phi$  is diagonalizable

$$\phi = m P^{-1} \text{diag}(y_1, \dots, y_N) P. \quad (2.29)$$

Then (2.28) reduces to a polynomial equation of order  $N + 1$  for  $x := e^{-\eta}$

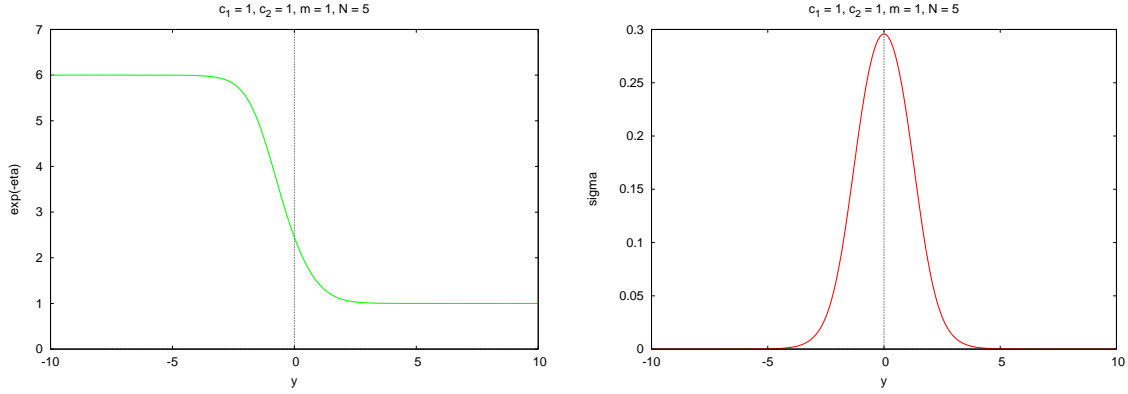
$$x = 1 + \frac{c_1}{c_2} \sum_{i=1}^N \frac{1}{1 + e_i x}, \quad e_i = e^{2m(y-y_i)}. \quad (2.30)$$

If this equation is solved, one can supply its solution into Eq.(2.27) to obtain  $\Omega$ .

In the simplest case, where all walls are coincident  $\phi = my_0 \mathbf{1}_N$ , we can solve equation (2.30) explicitly ( $e_0 := e^{2m(y-y_0)}$ ) to find

$$e^{-\eta} = \frac{1}{2e_0} \left( e_0 - 1 + \sqrt{(1 - e_0)^2 + 4(1 + Nc_1/c_2)e_0} \right), \quad (2.31)$$

$$\Omega = (e^{2my} + e^{2my_0} e^\eta) \mathbf{1}_N. \quad (2.32)$$



**Fig. 2** Profiles of  $e^{-\eta}$  and  $\sigma$  in the coincident case. The parameters of the plot are given above the picture. Positions of all domain walls are centered at the origin.

Physical fields can be expressed in terms of  $\Omega$  and  $\sigma$  as

$$H_1 = \sqrt{c_1} \frac{\mathbf{1}_N}{\sqrt{1 + e^{-2m(y-y_0)+\eta}}}, \quad (2.33)$$

$$H_2 = \sqrt{c_1} \frac{U^\dagger}{\sqrt{1 + e^{2m(y-y_0)-\eta}}}, \quad (2.34)$$

$$H_3 = \sqrt{c_2} e^{-\eta/2}, \quad (2.35)$$

$$\Sigma = \frac{1}{2} \partial_y \ln \Omega, \quad (2.36)$$

$$\sigma = \partial_y \eta, \quad (2.37)$$

$$W_y = A_y = 0, \quad (2.38)$$

where we fixed the gauge such that  $S = \Omega^{1/2}$  and  $\text{Im}(\psi) = 0$ .

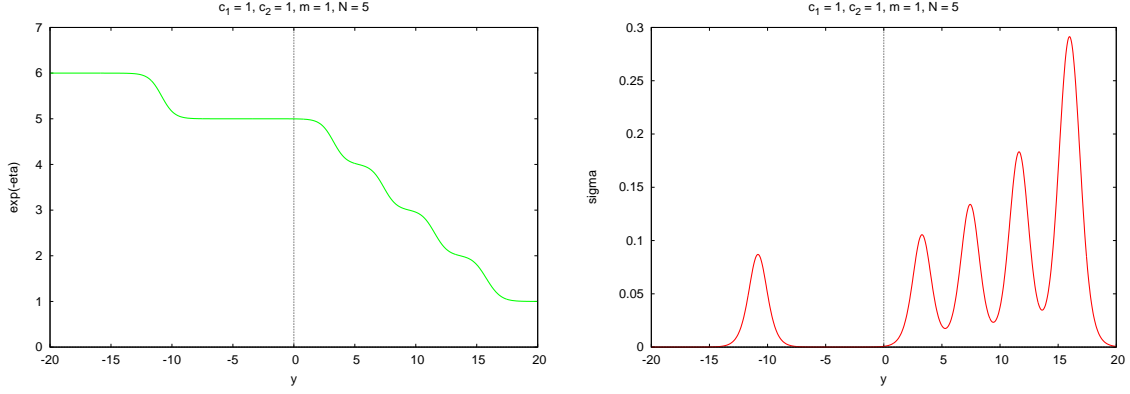
This set of solutions is not invariant under the symmetry transformations (2.4) in general. For a choice of  $U = \mathbf{1}_N$ , however, the solutions (2.33)-(2.38) are invariant under the action of the diagonal global symmetry  $SU(N)_{L+R+c}$ . We show the  $y$ -dependence of  $e^{-\eta}$  and of  $\sigma = \partial_y \eta/2$  for the coincident case in Fig.2.

For more general case such as non-coincident walls, the dependence of  $e^{-\eta}$  and  $\sigma$  on  $y$  is more complicated. Furthermore, the equation (2.30) cannot be solved in a closed form in general, except for first few values of  $N$ . Thus, one has to use numerical techniques. In Fig. 3, we present an example of five non-coincident walls.

## 2.2. Localization of non-Abelian gauge fields

In order to obtain massless gauge fields localized on the domain wall we need to introduce a new gauge symmetry which is not broken in the bulk. As we have seen in the previous subsections, the coincident domain wall solutions (2.33)-(2.38) do not break a large part of the global symmetry. Let us then gauge  $SU(N)_{L+R} \equiv SU(N)_V$  and denote new gauge fields as  $V_M$ . Then the fields  $H_1$  and  $H_2$  are in the bi-fundamental representation of  $SU(N)_c \times SU(N)_V$  and the covariant derivatives (2.3) are modified to

$$\tilde{D}_M H_{1,2} = \partial_M H_{1,2} + iW_M H_{1,2} - iH_{1,2} V_M. \quad (2.39)$$



**Fig. 3** Profiles of  $e^{-\eta}$  and  $\sigma$  in the non-coincident case. The parameters of the plot are given above the picture. Positions of domain walls are  $y_1 = -10, y_2 = 4, y_3 = 8, y_4 = 12$  and  $y_5 = 16$ .

|          | $SU(N)_c$                      | $U(1)_1$ | $U(1)_2$ | $SU(N)_V$    | $U(1)_A$ | mass            |
|----------|--------------------------------|----------|----------|--------------|----------|-----------------|
| $H_1$    | $\square$                      | 1        | 0        | $\square$    | 1        | $m\mathbf{1}_N$ |
| $H_2$    | $\square$                      | 1        | -1       | $\square$    | -1       | 0               |
| $H_3$    | $\mathbf{1}$                   | 0        | 1        | $\mathbf{1}$ | 0        | 0               |
| $\Sigma$ | $\text{adj} \oplus \mathbf{1}$ | 0        | 0        | $\mathbf{1}$ | 0        | 0               |
| $\sigma$ | $\mathbf{1}$                   | 0        | 0        | $\mathbf{1}$ | 0        | 0               |

**Table 2** Quantum numbers of the gauged model.

Quantum numbers of the gauged model are summarized in Tab. 2

We introduce field-dependent gauge coupling for  $V_M$  as

$$\frac{1}{2\tilde{g}^2(\sigma)} = \lambda\sigma, \quad (2.40)$$

where we assume that  $\lambda$  is a positive constant

$$\lambda > 0. \quad (2.41)$$

If we denote the field strength for the new gauge fields as  $\tilde{G}_{MN}$  the Lagrangian for the gauged model is given by

$$\mathcal{L} = \tilde{\mathcal{L}} - \frac{1}{2\tilde{g}^2(\sigma)} \text{Tr}[\tilde{G}_{MN}\tilde{G}^{MN}], \quad (2.42)$$

where  $\tilde{\mathcal{L}}$  is the same as in (2.1) except that the covariant derivatives (2.3) are replaced by (2.39). The choice of the field-dependent coupling (2.40) is inspired by supersymmetry. As discussed in Ref. [9], a term that is linear in adjoint scalars, appearing in front of the kinetic term for gauge field, naturally arises in five-dimensional  $\mathcal{N} = 1$  supersymmetric gauge theories [8]. In fact, our model can be embedded into  $\mathcal{N} = 1$  supersymmetry.

It is not hard to see that the solution (2.33)-(2.38) in the ungauged model is equally valid in the gauged model. If we write down the equation of motion for the new gauge field  $V_M$



we have

$$\partial_M \tilde{G}^{MN} = J^N, \quad (2.43)$$

where  $J_M$  stands for the current of  $V_M$ . Since the solution preserves  $SU(N)_V$ , the current vanishes for the domain wall solution (2.33)-(2.38), and  $V_M = 0$  is a valid solution to the equation of motion (2.43). Then the other equations of motion of the gauged model reduce to those of the ungauged model because of  $V_M = 0$ . Therefore, we see that (2.33)-(2.38) in addition to the condition  $V_M = 0$  solves the whole set of equations of motion of the gauged model.

### 2.3. Positivity of position-dependent gauge coupling

We wish to show that the field-dependent gauge coupling (2.40) assures the positive definiteness of the position-dependent gauge coupling for any configurations of the domain-wall. Since we do not need the effective theory in full, we reserve to derive the rest of the effective Lagrangian for the next section.

For the moment it is sufficient to know, that the field-dependent gauge coupling  $1/\tilde{g}^2(\sigma)$  is given by its value in the background solution

$$\left. \frac{1}{\tilde{g}^2(\sigma)} \right|_{\text{background}} \equiv \frac{1}{\tilde{g}^2(y)} = \lambda \partial_y \eta = -\lambda \partial_y \ln x, \quad (2.44)$$

where  $x = e^{-\eta} \geq 0$  is a solution to (2.30). Differentiating (2.30) we find

$$\frac{1}{x} \partial_y x = -\frac{c_1}{c_2} \sum_{i=1}^N \frac{e_i}{(1 + e_i x)^2} \left( 2m + \frac{1}{x} \partial_y x \right). \quad (2.45)$$

This leads to the formula

$$\frac{1}{2\tilde{g}^2(y)} = \frac{\lambda c_1}{c_2} \sum_{i=1}^N \frac{e_i}{(1 + e_i x)^2} \bigg/ \left( 1 + \frac{c_1}{c_2} \sum_{i=1}^N \frac{e_i}{(1 + e_i x)^2} \right), \quad (2.46)$$

which is indeed positive in the whole range of  $y$ -coordinate.

Integrating  $1/\tilde{g}^2(y)$  over the extra-dimensional coordinate  $y$  we obtain the effective gauge coupling in 3 + 1-dimensional world volume

$$\frac{1}{\tilde{g}^2} = \lambda \int_{-\infty}^{\infty} dy \partial_y \eta = \lambda [\eta(\infty) - \eta(-\infty)]. \quad (2.47)$$

The asymptotic values of  $\eta$  are found from (2.31) as

$$\eta(\infty) = 0, \quad \eta(-\infty) = -\ln \left( 1 + N \frac{c_1}{c_2} \right). \quad (2.48)$$

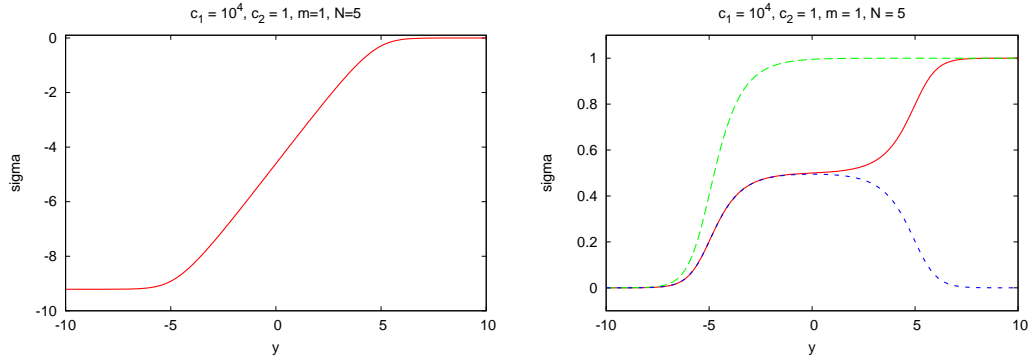
The easiest way to see these asymptotic values is to note that in (2.32) we can take the limits of (2.30) to obtain

$$x = 1 + \frac{c_1}{c_2} \sum_{i=1}^N \frac{1}{1 + e_i x} \longrightarrow \begin{cases} 1 & \text{if } y \rightarrow \infty, \\ 1 + \frac{N c_1}{c_2} & \text{if } y \rightarrow -\infty, \end{cases} \quad (2.49)$$

since  $\eta$  is finite at both infinities \*.

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\* We can just look at (2.19) with  $H_3^0 = \sqrt{c_2}$  and recall (2.6) to obtain the same result.



**Fig. 4** Profile of  $\eta$ -kink is shown in the left panel for the coincident case. In the right panel, plots of  $\text{Tr}[\Sigma]$  (green dashed curve),  $\text{Tr}[\Sigma] - \sigma$  (red solid curve), and  $\sigma$  (blue dotted curve) are shown.

Thus, the effective gauge coupling is given as

$$\frac{1}{\tilde{g}^2} = \lambda \ln \left( 1 + N \frac{c_1}{c_2} \right). \quad (2.50)$$

It is interesting to observe that the effective gauge coupling is proportional to the width of the domain wall  $\ln(1 + N c_1/c_2)$ , which is not a modulus, but is fixed by parameters of the theory. This feature is in sharp contrast to that in Ref. [10] where the effective gauge coupling constant is proportional to the domain wall width, which is a modulus undetermined by the theory. We have now confirmed the stability of the gauge kinetic term (by choosing the parameters of the theory as  $\lambda, m, c_1, c_2 > 0$ ).

Eq. (2.47) shows that the effective four-dimensional gauge coupling constant  $\tilde{g}$  is determined only by the boundary conditions at infinity. It can be interpreted as a kind of topological charge of the  $\eta$ -kink, whose profile is shown in Fig. 4.

Note that we have considered only the BPS solutions so far, and confirmed the stability of the model. The anti-BPS solutions are also stable, since they can be obtained by the parity transformation  $x \rightarrow -x$ .

### 3. Effective Lagrangian

In this section we calculate the low-energy effective Lagrangian on the background domain wall solution in the moduli approximation [16], where the moduli are promoted to fields on the world-volume with coordinates  $x^\mu$  and are assumed to depend only weakly on  $x^\mu$ . As the background we consider the coincident walls  $\phi_{bg} = m y_0 \mathbf{1}_N$ . Our model have two moduli: a Hermitian  $N \times N$  matrix  $\phi$  and a unitary  $N \times N$  matrix  $U$ . We study in two steps: taking only  $U$  as moduli fields, and then both  $\phi$  as well as  $U$ .

#### 3.1. Effective Lagrangian for coincident walls

Here we ignore the moduli fields  $\phi$ , and take only  $U$  moduli fields, that describe the Nambu-Goldstone modes associated with the symmetry breaking in Eq.(2.8). When  $\phi = m y_0 \mathbf{1}_N$ , the solution (2.17)-(2.19) with (2.23) and (2.24) become in the strong gauge coupling limit

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$g \rightarrow \infty$  and  $e \rightarrow \infty$  as

$$H_1 = \sqrt{c_1} e^{my} \Omega^{-1/2}, \quad (3.1)$$

$$H_2 = \sqrt{c_1} e^{\eta/2} e^{my_0(x)} \Omega^{-1/2} U^\dagger(x), \quad (3.2)$$

$$H_3 = \sqrt{c_2} e^{-\eta/2}, \quad (3.3)$$

where  $\Omega$  and  $\eta$  are defined in Eqs.(2.31) and (2.32) with the replacement  $y_0 \rightarrow y_0(x^\mu)$  and  $U \rightarrow U(x^\mu)$ . We plug these into the Lagrangian (2.42), where the gauge fields  $W_\mu$  and  $A_\mu$  are no longer dynamical and should be eliminated as auxiliary fields. After integrating over the extra-dimensional coordinate and taking up to quadratic terms in the derivatives, we obtain the low energy effective Lagrangian (the detailed calculation is given in Appendix A)

$$\begin{aligned} \mathcal{L}_{\text{eff}} = & \frac{c_1}{2m} \left[ (\alpha + 1) \text{Tr}[\mathcal{D}_\mu U^\dagger \mathcal{D}^\mu U] + \frac{\alpha}{N} \text{Tr}[U \mathcal{D}_\mu U^\dagger] \text{Tr}[U \mathcal{D}^\mu U^\dagger] \right] \\ & + \frac{N m c_1}{2} \partial_\mu y_0 \partial^\mu y_0 - \frac{1}{2m} \ln \left( 1 + \frac{N c_1}{c_2} \right) \text{Tr}[\tilde{G}_{\mu\nu} \tilde{G}^{\mu\nu}], \end{aligned} \quad (3.4)$$

where

$$\mathcal{D}_\mu = \partial_\mu U + i[V_\mu, U], \quad (3.5)$$

and

$$\alpha \equiv \frac{1}{2} + \frac{c_2}{N c_1} - \frac{c_2}{N c_1} \left( 1 + \frac{c_2}{N c_1} \right) \ln \left( 1 + \frac{N c_1}{c_2} \right). \quad (3.6)$$

Let us define the decay constants  $f_\pi$  for the adjoint field and  $f_\eta$  for the singlet field

$$f_\pi = \sqrt{\frac{c_1(\alpha + 1)}{2m}}, \quad f_\eta = \sqrt{\frac{c_1}{N m}}. \quad (3.7)$$

Then the canonically normalized adjoint and singlet fields can be defined as  $\hat{\pi}$  and  $\eta$  respectively

$$\frac{1}{f_\pi} \mathcal{D}_\mu \hat{\pi} = i \left[ U \mathcal{D}_\mu U^\dagger - \frac{\mathbf{1}_N}{N} \text{Tr}(U \mathcal{D}_\mu U^\dagger) \right], \quad (3.8)$$

$$\frac{1}{f_\eta} \partial_\mu \eta := i \text{Tr}(U \mathcal{D}_\mu U^\dagger). \quad (3.9)$$

The effective Lagrangian Eq. (3.4) can be rewritten as

$$\mathcal{L}_{\text{eff}} = \text{Tr} \left( \mathcal{D}_\mu \hat{\pi} \mathcal{D}^\mu \hat{\pi} \right) + \frac{1}{2} \partial_\mu \eta \partial^\mu \eta + \frac{N m c_1}{2} \partial_\mu y_0 \partial^\mu y_0 - \frac{1}{2 \tilde{g}^2} \text{Tr}[\tilde{G}_{\mu\nu} \tilde{G}^{\mu\nu}]. \quad (3.10)$$

A new feature of (3.10) compared with our previous work [10] is that the coupling strength  $f_\pi$  of the adjoint field is larger<sup>†</sup> than  $f_\eta$  of the singlet field by a factor  $\sqrt{\alpha + 1}$ .

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<sup>†</sup> Difference of factors of 2 and  $N$  is due to a convention of  $SU(N)$  matrix normalization  $\text{Tr}(T^I T^J) = \delta^{IJ}/2$ .

### 3.2. Full effective Lagrangian

As the final step, we consider the general case of both  $U$  and  $\phi$  as moduli fields. Then scalar fields are given in terms of moduli fields  $U(x^\mu)$  and  $\phi(x^\mu)$  as <sup>‡</sup>

$$H_1 = \sqrt{c_1} e^{my} \Omega^{-1/2}, \quad (3.11)$$

$$H_2 = \sqrt{c_1} e^{\eta/2} \Omega^{-1/2} e^\phi U^\dagger, \quad (3.12)$$

$$H_3 = \sqrt{c_2} e^{-\eta/2}, \quad (3.13)$$

$$\Omega^{-1} = \frac{e^{-\eta} e^{-2\phi}}{\mathbf{1}_N + e^{2my} e^{-2\phi} e^{-\eta}}, \quad (3.14)$$

$$e^{-\eta} = 1 + \frac{c_1}{c_2} \text{Tr} \left( \frac{\mathbf{1}_N}{\mathbf{1}_N + e^{2my} e^{-2\phi} e^{-\eta}} \right). \quad (3.15)$$

To obtain the effective Lagrangian, we need to repeat the same procedure as in the previous subsection, where the covariant derivatives acting on functions of matrices require more care (see e.g. Appendix B in Ref. [10]) and causes difficulty when deriving the closed form of the effective Lagrangian. However, we have a convenient parameter to expand the effective Lagrangian, the ratio  $c_1/c_2$  whose logarithm has a physical meaning as the width of the domain wall (see Eq.(2.50)). As given in Appendix A, the effective Lagrangian up to the order of  $c_1/c_2$  is given as

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{eff}}^{(0)} + \mathcal{T}_\phi^{(1)} + \mathcal{T}_U^{(1)} + \mathcal{T}_{mix}^{(1)} + \mathcal{T}'_\phi + \mathcal{T}'_U + c_1 O\left((c_1/c_2)^2\right), \quad (3.16)$$

where

$$\begin{aligned} \mathcal{L}_{\text{eff}}^{(0)} = & \frac{c_1}{2m} \text{Tr} \left[ \mathcal{D}_\mu \phi \frac{\cosh(\mathcal{L}_\phi) - 1}{\mathcal{L}_\phi^2 \sinh(\mathcal{L}_\phi)} \ln \left( \frac{1 + \tanh(\mathcal{L}_\phi)}{1 - \tanh(\mathcal{L}_\phi)} \right) (\mathcal{D}^\mu \phi) \right. \\ & + U^\dagger \mathcal{D}_\mu U \frac{\cosh(\mathcal{L}_\phi) - 1}{\mathcal{L}_\phi \sinh(\mathcal{L}_\phi)} \ln \left( \frac{1 + \tanh(\mathcal{L}_\phi)}{1 - \tanh(\mathcal{L}_\phi)} \right) (\mathcal{D}^\mu \phi) \\ & \left. + \frac{1}{2} \mathcal{D}_\mu U^\dagger U \frac{1}{\tanh(\mathcal{L}_\phi)} \ln \left( \frac{1 + \tanh(\mathcal{L}_\phi)}{1 - \tanh(\mathcal{L}_\phi)} \right) (U^\dagger \mathcal{D}^\mu U) \right], \end{aligned} \quad (3.17)$$

with a Lie derivative with respect to  $A$ ,  $\mathcal{L}_A(B) = [A, B]$ . Interestingly, this is the same effective Lagrangian we obtained previously [10]. The rest of terms are of order of  $c_1/c_2$ , given by

$$\mathcal{T}_U^{(1)} = \frac{c_1^2}{16c_2m} \text{Tr} \left[ \mathcal{D}_\mu U^\dagger U F_U(\partial_x, \mathcal{L}_\phi) (U^\dagger \mathcal{D}^\mu U) e^{x\phi} \right] \text{Tr} \left[ e^{-x\phi} \right] \Big|_{x=0}, \quad (3.18)$$

$$\mathcal{T}_{mix}^{(1)} = \frac{c_1^2}{8c_2m} \text{Tr} \left[ U^\dagger \mathcal{D}_\mu U F_{mix}(\partial_x, \mathcal{L}_\phi) (\mathcal{D}^\mu \phi) e^{x\phi} \right] \text{Tr} \left[ e^{-x\phi} \right] \Big|_{x=0}, \quad (3.19)$$

$$\mathcal{T}_\phi^{(1)} = \frac{c_1^2}{8c_2m} \text{Tr} \left[ \mathcal{D}_\mu \phi F_\phi(\partial_x, \mathcal{L}_\phi) (\mathcal{D}^\mu \phi) e^{x\phi} \right] \text{Tr} \left[ e^{-x\phi} \right] \Big|_{x=0}, \quad (3.20)$$

$$\mathcal{T}'_\phi = -\frac{c_1^2}{16c_2m} F(\partial_x) \text{Tr} \left[ e^{x\phi} \mathcal{D}_\mu \phi \right] \text{Tr} \left[ e^{-x\phi} \mathcal{D}^\mu \phi \right] \Big|_{x=0}, \quad (3.21)$$

$$\mathcal{T}'_U = \frac{c_1^2}{16c_2m} F(\partial_x) \text{Tr} \left[ e^{x\phi} \mathcal{D}_\mu U^\dagger U \right] \text{Tr} \left[ e^{-x\phi} \mathcal{D}^\mu U^\dagger U \right] \Big|_{x=0}, \quad (3.22)$$

<sup>‡</sup> The center of mass position of walls  $y_0(x^\mu)$  is a free field as in Eq.(3.10) and is suppressed here.

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and

$$F_U(x, \mathcal{L}_\phi) = \int_{-\infty}^{\infty} dy \frac{\cosh(\mathcal{L}_\phi)}{\cosh^2(y-x) \cosh(y) \cosh(y-\mathcal{L}_\phi)}, \quad (3.23)$$

$$F_{mix}(x, \mathcal{L}_\phi) = \int_{-\infty}^{\infty} dy \frac{\cosh(y) \cosh(y-\mathcal{L}_\phi) - \sinh(y) \sinh(y-\mathcal{L}_\phi) - 1}{\mathcal{L}_\phi \cosh(y) \cosh(y-\mathcal{L}_\phi) \cosh^2(y-x)}, \quad (3.24)$$

$$F_\phi(x, \mathcal{L}_\phi) = \int_{-\infty}^{\infty} dy \frac{\cosh(y) \cosh(y+\mathcal{L}_\phi) - \sinh(y) \sinh(y+\mathcal{L}_\phi) - 1}{\mathcal{L}_\phi^2 \cosh(y) \cosh(y+\mathcal{L}_\phi) \cosh^2(y-x)}, \quad (3.25)$$

$$F(x) = \int_{-\infty}^{\infty} dy \frac{1}{\cosh^2(y) \cosh^2(y-x)}. \quad (3.26)$$

All the above integrals can be obtained in closed forms.

#### 4. More general models of stable position-dependent coupling

In this section, we wish to show that there are more models with the stable position-dependent coupling. We will illustrate the point by extending the model to include more fields and more gauge symmetry.

The position-dependent gauge coupling comes from the cubic coupling between a singlet scalar field and field strengths of non-Abelian gauge fields in  $4+1$  dimensions

$$\mathcal{L}_{\text{cubic}} = C(\sigma_i) \text{Tr} \left[ \tilde{G}_{MN} \tilde{G}^{MN} \right], \quad (4.1)$$

where the function  $C(\sigma_i)$  of singlet scalar fields  $\sigma_i$  should be linear, if it is to be embeddable into a supersymmetric gauge theory in  $4+1$  dimensions

$$C(\sigma_i) = \sum_i \gamma_i \sigma_i, \quad \gamma_i \in \mathbb{R}, \quad (4.2)$$

where  $\gamma_i$  are constant coefficients.

Usually each domain wall has one complex moduli: a position and a phase. For example, both the massive  $\mathbb{CP}^2$  model and the massive  $\mathbb{CP}^1 \times \mathbb{CP}^1$  model have two free domain walls, corresponding to the two complex moduli. Although two free domain walls can produce the desired profile of position-dependent gauge coupling by an appropriate choice of parameters in Eq.(4.2), they provide the undesired modulus for the width of the profile. To avoid this problem, we are led to consider models with a single complex moduli. The simplest one of such models is the three flavor model in Ref.[9], where three scalar fields are constrained by the two Abelian gauge symmetry  $U(1) \times U(1)$ . Geometry of this three-flavor model is examined in Appendix B. Our model in this paper is an extension of this model to non-Abelian gauge group:  $U(1) \times U(1) \rightarrow U(N) \times U(1)$ . The next-simplest possibility is to consider four scalars constrained by three Abelian gauge symmetry  $U(1) \times U(1) \times U(1)$ , which we call four-flavor models. We give quantum numbers of fields of a typical four-flavor model in Tab.3. In the limit of strong gauge couplings, the gauge theory becomes a nonlinear sigma model whose target space is given by an intersection of three conditions as

$$(\mathbb{C}^2 \times \mathbb{CP}^1) \cap (\mathbb{H}^2 \times \mathbb{CP}^1) \cap (\mathbb{C}^2 \times \mathbb{CP}^1) \simeq \mathbb{CP}^1. \quad (4.3)$$

|            | $U(1)_1$ | $U(1)_2$ | $U(1)_3$ | mass  |
|------------|----------|----------|----------|-------|
| $H_1$      | 1        | 0        | 0        | $m_1$ |
| $H_2$      | 1        | -1       | 0        | $m_2$ |
| $H_3$      | 0        | 1        | 1        | $m_3$ |
| $H_4$      | 0        | 0        | 1        | $m_4$ |
| $\sigma_1$ | 0        | 0        | 0        | 0     |
| $\sigma_2$ | 0        | 0        | 0        | 0     |
| $\sigma_3$ | 0        | 0        | 0        | 0     |

**Table 3** Quantum numbers of the  $U(1)_1 \times U(1)_2 \times U(1)_3$  four-flavor model.

|                     | $ H_1 $            | $ H_2 $          | $ H_3 $          | $ H_4 $            | $\sigma_1$        | $\sigma_2$  | $\sigma_3$         |
|---------------------|--------------------|------------------|------------------|--------------------|-------------------|-------------|--------------------|
| $\langle 1 \rangle$ | 0                  | $\sqrt{c_1}$     | $\sqrt{c_{1+2}}$ | $\sqrt{c_{3-1-2}}$ | $m_2 + m_3 - m_4$ | $m_3 - m_4$ | $m_4$              |
| $\langle 2 \rangle$ | $\sqrt{c_1}$       | 0                | $\sqrt{c_2}$     | $\sqrt{c_{3-2}}$   | $m_1$             | $m_3 - m_4$ | $m_4$              |
| $\langle 3 \rangle$ | $\sqrt{c_{1+2}}$   | $\sqrt{-c_2}$    | 0                | $\sqrt{c_3}$       | $m_1$             | $m_1 - m_2$ | $m_4$              |
| $\langle 4 \rangle$ | $\sqrt{c_{1+2-3}}$ | $\sqrt{c_{3-2}}$ | $\sqrt{c_3}$     | 0                  | $m_1$             | $m_1 - m_2$ | $-m_1 + m_2 + m_3$ |

**Table 4** VEVs of candidate vacua: We use abbreviations like  $c_{3-1-2} \equiv c_3 - c_1 - c_2$ .

The vacuum condition is given by

$$|H_1|^2 + |H_2|^2 = c_1, \quad -|H_2|^2 + |H_3|^2 = c_2, \quad |H_3|^2 + |H_4|^2 = c_3, \quad (4.4)$$

$$H_1(\sigma_1 - m_1) = 0, \quad H_2(\sigma_1 - \sigma_2 - m_2) = 0, \quad (4.5)$$

$$H_3(\sigma_2 + \sigma_3 - m_3) = 0, \quad H_4(\sigma_3 - m_4) = 0, \quad (4.6)$$

where  $c_i$  is the Fayet-Iliopoulos parameter of  $U(1)_i$  and  $m_a$  is the mass for  $H_i$ . All possible solutions to these equations are shown in Tab. 4. There are four solutions but only two of them are valid solutions for any choice of real parameters of  $c_i$ . When we choose  $c_1 > 0$ ,  $c_2 > 0$  and  $c_3 > c_1 + c_2$ , we are left with the vacua  $\langle 1 \rangle$  and  $\langle 2 \rangle$  in Tab. 4.

The moduli matrix formalism [15, 17] is powerful enough to give generic solutions of the BPS equations of this nonlinear sigma model. Especially, we are interested in the kink profiles of  $\sigma_1$ ,  $\sigma_2$  and  $\sigma_3$ . They can be expressed as derivatives of real functions  $\eta_i$

$$\sigma_i = \frac{1}{2} \partial_y \eta_i, \quad (i = 1, 2, 3), \quad (4.7)$$

where the real functions  $\eta_i$  are determined by the following algebraic conditions

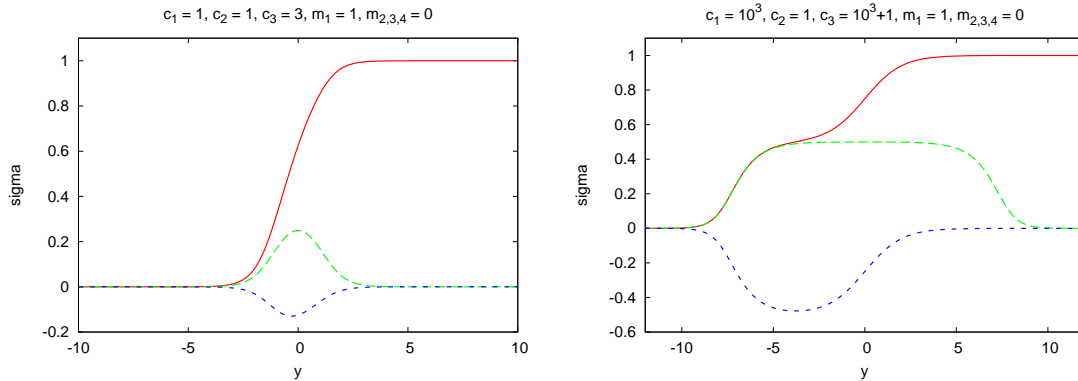
$$e^{-\eta_1 + 2m_1 y} + e^{-\eta_1 + \eta_2 + 2m_2 y - 2a} = c_1, \quad (4.8)$$

$$-e^{-\eta_1 + \eta_2 + 2m_2 y - 2a} + e^{-\eta_2 - \eta_3 + 2m_3 y} = c_2, \quad (4.9)$$

$$e^{-\eta_2 - \eta_3 + 2m_3 y} + e^{-\eta_3 + 2m_4 y} = c_3, \quad (4.10)$$

with a real constant  $a$ . The parameter  $a$  is (the real part of) the unique modulus of the solution, corresponding to the position of the domain wall. As we expected, we find only a single modulus. The width of the domain wall is not a modulus, but fixed by the theory.

Since we want a configuration that  $\sigma_i \rightarrow 0$  at both spacial infinities, we can choose  $m_3 = m_4 = 0$ . Then  $\sigma_2 = \sigma_3 \rightarrow 0$  at  $y = \pm\infty$ . Several numerical solutions are displayed in Fig. 5.



**Fig. 5** The kink profiles of  $\sigma_1$  (red solid line),  $\sigma_2$  (green dashed line) and  $\sigma_3$  (blue dotted line) for two different sets of model parameters.

From Fig. 5, we clearly see that  $\sigma_2 \geq 0$  and  $\sigma_3 \leq 0$  for all the values of  $y$ . As a position-dependent gauge coupling, we can choose a two-parameter family of desirable models

$$\mathcal{L}_{\text{CS}} = -(\gamma_2 \sigma_2 - \gamma_3 \sigma_3) \text{Tr} [\tilde{G}_{MN} \tilde{G}^{MN}], \quad (4.11)$$

where  $\gamma_{2,3}$  can be any non-negative real numbers. This class of models can be easily made to localize non-Abelian gauge fields and minimally interacting matter fields by extending two of the  $U(1)$  factor groups to (possibly different)  $U(N)$  gauge groups with  $N$  scalar fields in the fundamental representations, similarly to our model in previous sections. A new interesting feature of the four-flavor model (and its non-Abelian extensions) is that two possible profile of singlet fields  $\sigma_1, \sigma_2$  can provide a different profile for different non-Abelian gauge groups such as  $SU(3)$ ,  $SU(2)$  and  $U(1)$  with their associated matter fields.

## 5. Conclusion and discussion

We have successfully stabilized gauge fields and matter fields that are localized on domain walls. The low-energy effective Lagrangian on the domain wall has been worked out, where the adjoint matter fields is found to couple more strongly than singlet matter fields. We also explored possible generalization of our stabilization mechanism by including more fields and more gauge symmetries and found a class of more generic models with an added flexibility for model building with different localization profile for different gauge groups.

To build realistic models of brane-world with our scenario of localized gauge fields and matter fields, we should address several questions. Perhaps the most important question is to obtain (massless) matter fields in representations like the fundamental rather than the adjoint of localized gauge fields. One immediate possibility is to use the localization mechanism of fermions in a kink background. It has been found that zero modes of such fermions are localized in such a way to give automatically chiral fermions [18]. We will pursue this direction and associated anomaly questions further.

Secondly, we should devise a way to give small masses to our matter fields in order to do phenomenology. Since some of our matter fields are the Nambu-Goldstone modes of a broken global symmetry, we need to consider an explicit breaking of such global symmetry.

Thirdly, another question is to study possibility of supersymmetric model of gauge field localization. We need to settle the issues of possible new moduli in that case. Moreover, we should examine the mechanism of supersymmetry breaking.

Finally it is an interesting possibility for model building to localize gauge fields of different gauge groups with different profiles. This situation is often proposed in recent brane-world phenomenology, for instance in Ref.[19].

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## A. Derivation of the effective Lagrangian

In this appendix, we derive the effective Lagrangians (3.4) and (3.16) in the strong coupling limit. Since we are interested in the low energy effective Lagrangian, we focus on up the quadratic terms in derivatives, which we call  $\mathcal{L}^{(2)}$ .

### A.1. Eliminating gauge fields

Here we eliminate gauge fields  $W_\mu, A_\mu$  to obtain a nonlinear sigma model. Starting with  $\tilde{\mathcal{L}}$  in (2.42) in the strong gauge coupling limit  $g \rightarrow \infty$  and  $e \rightarrow \infty$ ,  $\mathcal{L}^{(2)}$  is given as

$$\mathcal{L}^{(2)} = \text{Tr} |D_\mu H_1|^2 + \text{Tr} |(D_\mu - iA_\mu)H_2|^2 + |(\partial_\mu + iA_\mu)H_3|^2, \quad (\text{A1})$$

where the covariant derivatives are given as:

$$D_\mu H_{1,2} = \hat{D}_\mu H_{1,2} + iW_\mu H_{1,2} = \partial_\mu H_{1,2} - iH_{1,2}V_\mu + iW_\mu H_{1,2}. \quad (\text{A2})$$

Here we singled-out covariant derivatives  $\hat{D}_\mu$  containing  $V_\mu$  fields associated with the gauged part of the flavor symmetry  $SU(N)_{L+R}$ . Notice that  $W_\mu$  and  $A_\mu$  are no longer dynamical, but they are merely auxiliary fields to be eliminated. Moreover, the constraints are satisfied by the scalar fields (2.33)-(2.35) which depend on the moduli fields  $\phi(x^\mu)$  and  $U(x^\mu)$

$$H_1 H_1^\dagger + H_2 H_2^\dagger = c_1 \mathbf{1}_N, \quad (\text{A3})$$

$$|H_3|^2 - \text{Tr}(H_2 H_2^\dagger) = c_2. \quad (\text{A4})$$

The equation of motion for  $W_\mu$  gives

$$W_\mu = -\frac{i}{2c_1} \left[ H_a \hat{D}_\mu H_a^\dagger - \hat{D}_\mu H_a H_a^\dagger \right] + \frac{1}{c_1} A_\mu H_2 H_2^\dagger \equiv \hat{W}_\mu + \frac{1}{c_1} A_\mu H_2 H_2^\dagger, \quad (\text{A5})$$



where the sum over the index  $a = 1, 2$  is implied (Einstein summation convention). Plugging this back into (A1) we obtain

$$\begin{aligned}\mathcal{L}^{(2)} = & \text{Tr}(\hat{D}_\mu H_a \hat{D}^\mu H_a^\dagger) - c_1 \text{Tr}(\hat{W}_\mu \hat{W}^\mu) + i A_\mu \left( H_3 \partial^\mu H_3^\dagger - \partial^\mu H_3 H_3^\dagger \right. \\ & \left. - \text{Tr}(H_2 \hat{D}^\mu H_2^\dagger - \hat{D}^\mu H_2 H_2^\dagger) \right) - 2 A_\mu \text{Tr}(\hat{W}^\mu H_2 H_2^\dagger) \\ & + A_\mu A^\mu \left( |H_3|^2 + \text{Tr}(H_2 H_2^\dagger) - \frac{1}{c_1} \text{Tr}[(H_2 H_2^\dagger)^2] \right) + \partial_\mu H_3^\dagger \partial^\mu H_3. \quad (\text{A6})\end{aligned}$$

Next step is to eliminate the auxiliary fields  $A_\mu$ . By using the following identities derived from the constraint (A3)

$$\text{Tr}[(H_2 H_2^\dagger)^2] = c_1 \text{Tr}(H_2 H_2^\dagger) - \text{Tr}(H_2 H_2^\dagger H_1 H_1^\dagger), \quad (\text{A7})$$

$$\begin{aligned}-2 \text{Tr}(\hat{W}^\mu H_2 H_2^\dagger) = & i \text{Tr}(H_2 \hat{D}_\mu H_2^\dagger - \hat{D}_\mu H_2 H_2^\dagger) \\ & + \frac{i}{c_1} \epsilon_{ab} \text{Tr}[(H_a \hat{D}_\mu H_a^\dagger - \hat{D}_\mu H_a H_a^\dagger) H_b H_b^\dagger], \quad (\text{A8})\end{aligned}$$

and by solving equations of motion for  $A_\mu$ , we obtain

$$A_\mu = -\frac{i}{2} \frac{H_3 \partial_\mu H_3^\dagger - \partial_\mu H_3 H_3^\dagger + \frac{1}{c_1} \epsilon_{ab} \text{Tr}[(H_a \hat{D}_\mu H_a^\dagger - \hat{D}_\mu H_a H_a^\dagger) H_b H_b^\dagger]}{|H_3|^2 + \frac{1}{c_1} \text{Tr}(H_2 H_2^\dagger H_1 H_1^\dagger)}. \quad (\text{A9})$$

By using the following identities

$$\text{Tr}[\hat{D}_\mu H_a \hat{D}^\mu H_a^\dagger - c_1 \hat{W}_\mu \hat{W}^\mu] = \frac{1}{2c_1} \text{Tr}[\mathcal{D}_\mu \mathcal{H}_{ab} \mathcal{D}^\mu \mathcal{H}_{ab}^\dagger], \quad (\text{A10})$$

where  $\mathcal{H}_{ab} := H_a^\dagger H_b$  and  $\mathcal{D}_\mu \mathcal{H}_{ab} = \partial_\mu \mathcal{H}_{ab} + i[V_\mu, \mathcal{H}_{ab}]$ , and

$$\epsilon_{ab} \text{Tr}[(H_a \hat{D}_\mu H_a^\dagger - \hat{D}_\mu H_a H_a^\dagger) H_b H_b^\dagger] = \text{Tr}[\mathcal{H}_{12}^\dagger \mathcal{D}_\mu \mathcal{H}_{12} - \mathcal{D}_\mu (\mathcal{H}_{12}^\dagger) \mathcal{H}_{12}], \quad (\text{A11})$$

we obtain a simpler expression for  $A_\mu$

$$A_\mu = -\frac{i}{2} \frac{H_3 \partial_\mu H_3^\dagger - \partial_\mu H_3 H_3^\dagger + \frac{1}{c_1} \text{Tr}[\mathcal{H}_{12}^\dagger \mathcal{D}_\mu \mathcal{H}_{12} - \mathcal{D}_\mu (\mathcal{H}_{12}^\dagger) \mathcal{H}_{12}]}{|H_3|^2 + \frac{1}{c_1} \text{Tr}[\mathcal{H}_{12}^\dagger \mathcal{H}_{12}]}. \quad (\text{A12})$$

Using (A12), we can rewrite the effective Lagrangian as an integral of a nonlinear sigma model over  $y$

$$\begin{aligned}\mathcal{L}_{\text{eff}} = & \int_{-\infty}^{\infty} dy \left[ \frac{1}{2c_1} \text{Tr}[\mathcal{D}_\mu \mathcal{H}_{ab} \mathcal{D}^\mu \mathcal{H}_{ab}^\dagger] + \partial_\mu H_3 \partial^\mu H_3^\dagger \right. \\ & \left. - A_\mu A^\mu \left( |H_3|^2 + \frac{1}{c_1} \text{Tr}(\mathcal{H}_{12}^\dagger \mathcal{H}_{12}) \right) \right]. \quad (\text{A13})\end{aligned}$$

## A.2. Effective Lagrangian for $U$ and $y_0$

Let us now calculate the effective Lagrangian including only fluctuations  $U$  and  $y_0$  around the coincident domain wall solutions (3.1)-(3.3) with (2.31) and (2.32). The composite fields

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$\mathcal{H}_{ab}$  are given as ( $e_0 \equiv e^{m(y-y_0)}$ )

$$\mathcal{H}_{11} = c_1 \frac{e_0 e^{-\eta}}{1 + e_0 e^{-\eta}} \mathbf{1}_N, \quad (\text{A14})$$

$$\mathcal{H}_{12} = c_1 \frac{e_0^{1/2} e^{-\eta/2}}{1 + e_0 e^{-\eta}} U^\dagger, \quad (\text{A15})$$

$$\mathcal{H}_{22} = c_1 \frac{1}{1 + e_0 e^{-\eta}} \mathbf{1}_N. \quad (\text{A16})$$

After some algebra, covariant derivatives of these can be rewritten as

$$\mathcal{D}_\mu \mathcal{H}_{11} = -\frac{2c_2}{N} e^{-\eta} \sigma \partial_\mu y_0 \mathbf{1}_N, \quad (\text{A17})$$

$$\begin{aligned} \mathcal{D}_\mu \mathcal{H}_{12} &= \frac{2c_2}{N} e^{-\eta} \sigma \sinh(m(y - y_0) - \eta/2) \partial_\mu y_0 U^\dagger \\ &\quad + \frac{c_1}{2} \frac{\mathcal{D}_\mu U^\dagger}{\cosh(m(y - y_0) - \eta/2)}, \end{aligned} \quad (\text{A18})$$

$$\mathcal{D}_\mu \mathcal{H}_{22} = \frac{2c_2}{N} e^{-\eta} \sigma \partial_\mu y_0 \mathbf{1}_N, \quad (\text{A19})$$

where

$$\frac{\sigma}{m} = \left( 1 + \frac{c_2}{N c_1} (1 + e_0 e^{-\eta})^2 / e_0 \right)^{-1}. \quad (\text{A20})$$

Putting this into Eq. (A12) we obtain:

$$A_\mu = -\frac{i\sigma}{2Nm} \text{Tr}[U \mathcal{D}_\mu U^\dagger - U^\dagger \mathcal{D}_\mu U]. \quad (\text{A21})$$

Substituting this result into (A13), we obtain

$$\begin{aligned} \mathcal{L}_{\text{eff}} &= \frac{c_2}{Nm} \int_{-\infty}^{\infty} dy \frac{\sigma}{1 - \sigma/m} e^{-\eta} \text{Tr}[\mathcal{D}_\mu U^\dagger \mathcal{D}^\mu U] + m c_2 \int_{-\infty}^{\infty} dy e^{-\eta} \sigma \partial_\mu y_0 \partial^\mu y_0 \\ &\quad + \frac{c_2}{N^2 m^2} \int_{-\infty}^{\infty} dy \frac{\sigma^2}{1 - \sigma/m} e^{-\eta} \text{Tr}[U \mathcal{D}_\mu U^\dagger] \text{Tr}[U \mathcal{D}^\mu U^\dagger]. \end{aligned} \quad (\text{A22})$$

Finally integration over  $y$  is most easily done by substituting  $x = e^{-\eta}$  and using the identity:

$$\frac{\sigma}{m} = \frac{(x - 1) \left( 1 - \frac{c_2}{N c_1} (x - 1) \right)}{x + (x - 1) \left( 1 - \frac{c_2}{N c_1} (x - 1) \right)}, \quad (\text{A23})$$

leading to Eq.(3.4) (including the kinetic term for the gauge fields).

### A.3. Effective Lagrangian for all terms: $\phi$ , $U$ and $y_0$

Next we derive the effective Lagrangian including all fluctuations. Plugging the solution (3.11)-(3.13) into the Lagrangian (A13), we obtain

$$\begin{aligned}\mathcal{L}_{\text{eff}} = & \int_{-\infty}^{\infty} dy \left[ \mathcal{T}_U + \mathcal{T}_{\text{mix}} + \mathcal{T}_{\phi} \right] \\ & + \frac{c_1^2}{16c_2} \int_{-\infty}^{\infty} dy e^{\eta} \left( 1 - \frac{\sigma}{m} \right) \text{Tr} \left[ \frac{1}{\cosh^2(\hat{y})} \mathcal{D}_{\mu} U^{\dagger} U \right] \text{Tr} \left[ \frac{1}{\cosh^2(\hat{y})} \mathcal{D}^{\mu} U^{\dagger} U \right] \\ & - \frac{c_1^2}{16c_2} \int_{-\infty}^{\infty} dy e^{\eta} \left( 1 - \frac{\sigma}{m} \right) \text{Tr} \left[ \frac{1}{\cosh^2(\hat{y})} \mathcal{D}_{\mu} \phi \right] \text{Tr} \left[ \frac{1}{\cosh^2(\hat{y})} \mathcal{D}^{\mu} \phi \right],\end{aligned}\quad (\text{A24})$$

where

$$\begin{aligned}\mathcal{T}_U = & \frac{c_1}{4} \text{Tr} \left\{ \mathcal{D}_{\mu} U^{\dagger} \mathcal{D}^{\mu} U \frac{1}{\cosh^2(\hat{y})} \right. \\ & \left. + \mathcal{D}_{\mu} U^{\dagger} U \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \mathcal{L}_{\phi}^n (U^{\dagger} \mathcal{D}^{\mu} U) \left( \frac{e^{\hat{y}}}{\cosh(\hat{y})} \right)^{(n)} \frac{e^{-\hat{y}}}{\cosh(\hat{y})} \right\},\end{aligned}\quad (\text{A25})$$

$$\begin{aligned}\mathcal{T}_{\text{mix}} = & -\frac{c_1}{2} \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \text{Tr} \left\{ U^{\dagger} \mathcal{D}_{\mu} U \mathcal{L}_{\phi}^{n-1} (\mathcal{D}^{\mu} \phi) \left[ \left( \frac{1}{\cosh(\hat{y})} \right)^{(n)} \frac{1}{\cosh(\hat{y})} \right. \right. \\ & \left. \left. + \left( \tanh(\hat{y}) \right)^{(n)} \tanh(\hat{y}) \right] \right\},\end{aligned}\quad (\text{A26})$$

$$\begin{aligned}\mathcal{T}_{\phi} = & \frac{c_2}{4} \frac{\sigma/m - 2}{1 - \sigma/m} e^{-\eta} \partial_{\mu} \eta \partial^{\mu} \eta + \frac{c_1}{4} \text{Tr} \left[ \frac{1}{\cosh^2(\hat{y})} \mathcal{D}_{\mu} \phi \mathcal{D}^{\mu} \phi \right] \\ & - \frac{c_1}{2} \sum_{n=3}^{\infty} \frac{1}{n!} \text{Tr} \left\{ \mathcal{L}_{\phi}^{n-2} (\mathcal{D}_{\mu} \phi) \mathcal{D}^{\mu} \phi \left[ \left( \frac{1}{\cosh(\hat{y})} \right)^{(n)} \frac{1}{\cosh(\hat{y})} \right. \right. \\ & \left. \left. + \left( \tanh(\hat{y}) \right)^{(n)} \tanh(\hat{y}) \right] \right\},\end{aligned}\quad (\text{A27})$$

and

$$\hat{y} = (my - \eta/2) \mathbf{1}_N - \phi. \quad (\text{A28})$$

The only work which remains to be done is to perform the integration. This, however, turns out to be difficult. The main source of difficulty lies in the fact the we cannot solve (3.15) explicitly. We therefore use some approximation technique, such as the Taylor expansion. We find it convenient to use as an expansion parameter  $c_1/c_2$  which determines the width of the coincident wall. In the lowest order of approximation we see that Eq.(3.15) reduces to  $e^{-\eta} = 1$  and we obtain the effective Lagrangian in Eq.(3.16).

### A.4. Complexity of the Effective Lagrangian

The formula (3.16) illustrates the complexity of the interactions between moduli fields  $\phi$  and  $U$  in the general case. Let us offer some explanation for this complexity. The structure,

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|            | $U(1)_1$ | $U(1)_2$ | mass |
|------------|----------|----------|------|
| $H_1$      | 1        | 0        | $m$  |
| $H_2$      | 1        | 1        | 0    |
| $H_3$      | 0        | -1       | 0    |
| $\sigma_1$ | 0        | 0        | 0    |
| $\sigma_2$ | 0        | 0        | 0    |

**Table B1** Quantum numbers of the  $U(1)_1 \times U(1)_2$  three-flavor model.

which is new in (3.16) compared to (3.17) is of the general form

$$F(\partial_x) \text{Tr}[e^{x\phi} M(y)] \text{Tr}[e^{-x\phi} N(y)] \Big|_{x=0}, \quad (\text{A29})$$

where  $M(y)$  and  $N(y)$  are some matrix-valued functions, containing either derivatives of  $\phi$  or derivatives of  $U$ . If we assume that  $\phi = mP^{-1} \text{diag}(y_1, \dots, y_N)P$  is diagonalizable we can rewrite the above as

$$\sum_{i,j=1}^N F(m(y_i - y_j)) (PM(y)P^{-1})_{ii} (PN(y)P^{-1})_{jj}. \quad (\text{A30})$$

This form suggests that interaction (here represented by function  $F$ ) depends on the relative size of fluctuation of each pairs of walls. Indeed, notice that if two walls have the same position  $y_i = y_j$   $i \neq j$  (meaning that the expectation values of the fluctuations is the same), the above form reduces to

$$F(0) \text{Tr}(M) \text{Tr}(N), \quad (\text{A31})$$

which is in a sense trivial, since we already encountered this kind of terms in (3.4). Thus, the new kind of complexity in our result (3.16) can be understood as a manifestation of the fact, that the interaction does not depend only on various moments of the fluctuation as in (3.17) but also on their relative size.

## B. Geometry of the three-flavor model

Quantum numbers of three-flavor model are shown in Tab. B1. Since we are interested in the domain wall solutions, it is enough to consider the strong gauge coupling limit where gauge theories become non-linear sigma models (NLSM), whose target space is defined as an intersection of two spaces as

$$(\mathbb{C} \times \mathbb{C}P^1) \cap (\mathbb{C} \times \mathbb{H}^2) \simeq \mathbb{C}P^1. \quad (\text{B1})$$

Here  $\mathbb{H}^2$  stands for the two dimensional hyperbolic plane. In the above expression, The  $\mathbb{C}P^1$  and  $\mathbb{H}^2$  are defined by

$$\mathbb{C}P^1 = \{(H_1, H_2) \mid |H_1|^2 + |H_2|^2 = c_1, H_{1,2} \in \mathbb{C}\} / U(1)_1, \quad (\text{B2})$$

$$\mathbb{H}^2 = \{(H_2, H_3) \mid |H_2|^2 - |H_3|^2 = -c_2, H_{2,3} \in \mathbb{C}\} / U(1)_2. \quad (\text{B3})$$

This space is isomorphic to  $\mathbb{C}P^1$  but it is a squashed sphere. The metric can be read from the Kähler potential

$$K = e^{-V_1} |H_1|^2 + e^{-V_1 - V_2} |H_2|^2 + e^{V_2} |H_3|^2 + c_1 V_1 - c_2 V_2. \quad (\text{B4})$$

One can eliminate the real superfields  $V_1$  and  $V_2$ , and find the following expression of the Kähler potential with respect to the gauge invariant inhomogeneous coordinate  $\varphi$  as

$$K = c_1 [f + \log(|\varphi|^{-2} + f^{-1}) - \lambda \log f], \quad (\text{B5})$$

$$f = \frac{1}{2} \left( \lambda - |\varphi|^2 + \sqrt{4|\varphi|^2(1 + \lambda) + (\lambda - |\varphi|^2)^2} \right), \quad (\text{B6})$$

$$\varphi = \frac{H_2 H_3}{c_1 H_1}, \quad \lambda \equiv \frac{c_2}{c_1}. \quad (\text{B7})$$

Note that this manifold is singular at  $\varphi = 0$  when  $c_2 = 0$ . This can be understood in two different ways. The first one is to realize the fact that  $U(1)_2$  gauge symmetry is restored at that point. This is because  $|H_2| = |H_3| = 0$  holds there. Namely, the additional massless degrees of freedom should be taken into account. The second way is more straightforward. Let us calculate the scalar curvature in the vicinity of  $\varphi = 0$ . To this end, first we change the coordinate by  $\varphi = e^{i\Phi} \tan \frac{\Theta}{2}$  ( $0 \leq \Theta \leq \pi$ ,  $0 \leq \Phi \leq 2\pi$ ), then we obtain

$$R = \frac{8(8 + 9\lambda)}{9} \frac{1}{c_2} + \mathcal{O}(\Theta^2). \quad (\text{B8})$$

From this it is clear that the scalar curvature at  $\varphi = 0$  becomes infinity when  $c_2 = 0$ .

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